Distributional and Classical Solutions to the Cauchy Boltzmann Problem for Soft Potentials with Integrable Angular Cross Section

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Abstract This paper focuses on the study of existence and uniqueness of distributional and classical solutions to the Cauchy Boltzmann problem for the soft potential case assuming S^{n-1} integrability of the angular part of the collision kernel (Grad cut-off assumption). For this purpose we revisit the Kaniel–Shinbrot iteration technique to present an elementary proof of existence and uniqueness results that includes the large data near local Maxwellian regime with possibly infinite initial mass. We study the propagation of regularity using a recent estimate for the positive collision operator given in (Alonso et al. in Convolution inequalities for the Boltzmann collision operator. arXiv:0902.0507 [math.AP]), by E. Carneiro and the authors, that allows us to show such propagation without additional conditions on the collision kernel. Finally, an L^p-stability result (with $1 \le p \le \infty$) is presented assuming the aforementioned condition.

Keywords Boltzmann equation for soft potentials · Generalized and classical solutions · Stability in L^p spaces

1 Introduction

The purpose of this work is to study the standard model in the kinetic theory of gases given by the Boltzmann Transport Equation in the particular case of soft potentials (i.e. collision kernels with singular forms of the relative speed) under the S^{n-1} integrability assumption in the angular part of the collision kernel (Grad cut-off assumption).

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More specifically, we study the classical Cauchy Boltzmann problem: Find a function $f(t, x, v) \geq 0$ that solves the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f) \quad \text{in } (0, +\infty) \times \mathbb{R}^{2n} \tag{1.1}$$

and with initial condition $f(0, x, v) = f_0(x, v)$.

Much of the difficulty in solving this problem comes from the complexity of the collision operator Q which is defined for any two (suitable) functions f and g by the expression

$$Q(f, g) := \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ f('v)g('v_*) - f(v)g(v_*) \right\} B(|u|, \hat{u} \cdot \sigma) \, d\sigma \, dv_*, \tag{1.2}$$

where the symbols ${}'v$, ${}'v_*$, u are defined by

$$'v = v - (u \cdot \sigma)\sigma, \qquad 'v_* = v_* + (u \cdot \sigma)\sigma \quad \text{and} \quad u = v - v_*.$$

In addition, we use $\hat{x}$ to denote the unitary vector in the direction of x, for $x \in \mathbb{R}^n$. The function $B \geq 0$ is known as cross section collisional kernel and depends on the type of interaction between the particles of the gas. It is usually model as

$$B(|u|, \hat{u} \cdot \sigma) = \mathcal{G}(|u|)b(\hat{u} \cdot \sigma),$$

where $\mathcal{G}(|u|)$ is specified below and $b(\hat{u} \cdot \sigma)$ is S^{n-1}-integrable with respect to σ.

For a detailed presentation on the physical meaning of (1.1) see Ref. [8].

Solutions to this problem are known to exists in the *renormalized sense* for initial data with finite mass, energy and entropy. We refer to [9] for the study of renormalized solutions and to [1, 17, 20, 26] for further developments in the theory of very weak solutions.

Existence of distributional solutions (i.e. mild or weak solutions) when the initial data is "small" in some sense, or when is locally "close" to the equilibrium (Maxwell distribution) is also well documented. The theory of distributional solutions for the inhomogeneous Boltzmann case for small initial data started in the early 80's with the work of Illner and Shinbrot [18] who adapted the technique proposed in [19] to produce global in time solutions in the hard potential case. More recent developments on this theory, including the treatment of the soft potential case, can be found in [5, 16, 23].

There is also some work in the near local equilibrium case (i.e. near a local Maxwellian distribution). We refer to [22] for a treatment for the existence and uniqueness of distributional solutions in the case of hard spheres. Meanwhile in Refs. [11, 24] the soft potential case is discussed including the trend to equilibrium.

In the near global equilibrium case (i.e near a global Maxwellian distribution), the literature is focused on the existence and uniqueness of classical solutions in the soft potential case. The near global equilibrium theory greatly differs from the previous perturbative theories, such as near vacuum and near local Maxwellian, and was started with independent works of Caflisch [7] and Ukai-Asano [25] in early 80's. We refer to [12] for recent developments using energy methods to prove existence and uniqueness of classical solutions in the full soft potential range.

Regarding stability of solutions, Ha has shown the L^1-stability of solutions, see [14, 15], in regimes near local Maxwellian distributions as well as near vacuum. His results range soft and hard potentials and are valid for the pointwise cutoff condition $b(\hat{u} \cdot \sigma) \leq K \cos(\hat{u} \cdot \sigma)$.

It is worth to mention that the existence of distributional (mild or weak) solutions in the full soft potential range is an open problem. This problem as well as the propagation

of regularity to produce a global in time estimate on the spatial gradient of solutions and the L^p-stability, with $1 \leq p \leq \infty$, are the problems treated in this manuscript for initial states near local Maxwellians or vacuum states, all in the case without pointwise cut-off or regularization in the collision kernel.

More specifically, we treat both the distributional and classical theory in a more unified way for the cases of near vacuum and near local Maxwellian distributions. We will use techniques that simplify existing proofs and generalize the assumptions on the collision kernel, i.e. avoiding pointwise cut-off or regularization, especially when obtaining classical solutions. In addition, we reduce the smoothness assumptions on the initial data in order to obtain smooth classical solutions in both the small data and near local equilibrium cases maintaining minimal regularity on the collision kernel. As for stability of solutions, we prove under the Grad's cutoff assumption the L^p-stability ($1 \leq p \leq \infty$) of the solutions found in the aforementioned cases. The result presented will hold for soft potentials and Maxwell molecules.

This paper is organized as follows: In Sect. 2, distributional solutions for the Cauchy Boltzmann problem are constructed under the condition of "small" initial data. A brief discussion of Kaniel-Shinbrot iteration is presented before the main result is proved. This result of existence and uniqueness applies for both soft and hard potentials. For the remainder of the paper the proof will be focused only in the soft potential case, thus, in Sect. 3 solutions for the Boltzmann Cauchy problem are constructed for the near equilibrium case. More specifically, they are build under the assumption that the initial data is "close" to the local Maxwellian ($0 < \alpha, 0 \leq \beta$)

$$\exp(-\alpha |x - v|^2 - \beta |v|^2). \tag{1.3}$$

The main idea of this discussion is taken from Ref. [24] and, as in Sect. 2, uses the Kaniel-Shinbrot iteration approach. We construct a lower and upper Maxwellian distribution barriers to implement the iteration. Although our methodology is similar to that of [24], we have a more direct approach that leads to a relaxation on the conditions imposed to the barriers, in particular, we permit lower and upper barrier with different decay toward infinity. We note that Ref. [11] also allows different decay for the lower and upper Maxwellian distribution barriers using a functional fixed point argument. We follow the methodology given in [24] since it is more direct and no additional proof is required to obtain uniqueness of solutions.

In Sect. 4, we study the L^p-propagation of the solution's gradient which permits to construct classical solutions of (1.1) with natural regularity assumptions on the initial datum, in particular, we assume $\nabla_x f_0 \in L^p(\mathbb{R}^{2n})$ for some $1 < p < \infty$. We use a new estimate developed by Carneiro and the authors [3, 4] that allows to study the propagation of regularity without pointwise cut-off or regularization in the collision kernel, and to produce a global in time estimate on the spatial gradient of solutions. In this way we generalize some of the aspects treated in [6]. Furthermore, we address the propagation of regularity in the velocity variable by giving a global in time estimate in the velocity gradient.

Finally, two elementary results on stability are shown: stability of solutions in the space of functions uniformly bounded by Maxwellian distributions in the near vacuum case, and a L^p-stability of solutions ($1 \leq p \leq \infty$) in both cases. The latter is proved under the Grad's cutoff condition, see (A.2) below, on the angular kernel using the techniques previously presented in that section.

1.1 Assumption on the Model

Assume that the collision kernel $B(|u|, \hat{u} \cdot \sigma)$ satisfies

(A.1) $B(|u|, \hat{u} \cdot \sigma) = |u|^{-\lambda} b(\hat{u} \cdot \sigma)$ with $0 \le \lambda < n - 1$.

(A.2) Grad's assumption: $b(\hat{u} \cdot \sigma) \in L^1(S^{n-1})$. We will denote

$$\|b\|_{L^1(S^{n-1})} = \int_{S^{n-1}} b(\hat{u} \cdot \sigma) \, d\sigma.$$

Grad's assumption allows to split the collision operator in a gain and a loss part, namely,

$$Q(f, g) = Q_+(f, g) - Q_-(f, g)$$

with obvious definitions for each part. Moreover, the negative part can be expressed as

$$Q_-(f, g) = f R(g),$$

where

$$R(g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} g(v_*) |u|^{-\lambda} b(\hat{u} \cdot \sigma) \, d\sigma \, dv_*$$

$$= \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} g(v_*) |u|^{-\lambda} \, dv_* = \|b\|_{L^1(S^{n-1})} g * |v|^{-\lambda}. \tag{1.4}$$

1.2 Notation and Spaces

We first introduce some functional spaces where we seek and study solutions to our problem.

Let $M_{\alpha,\beta}(x, v) := \exp(-\alpha|x|^2 - \beta|v|^2)$ and define the space of functions bounded by a space-velocity Maxwellian, denoted by $\mathcal{M}_{\alpha,\beta}$, as

$$\mathcal{M}_{\alpha,\beta} = L^\infty\big(\mathbb{R}^{2n}, M_{\alpha,\beta}^{-1}\big),$$

endowed with the norm

$$\|f\|_{\alpha,\beta} = \big\| f M_{\alpha,\beta}^{-1} \big\|_{L^\infty(\mathbb{R}^{2n})}.$$

Let X be a Banach space. Define $W^{1,1}(0, T; X)$ as those functions $f \in L^1(0, T; X)$ such that its time derivative f_t exists in the weak sense and belongs to $L^1(0, T; X)$. This space is endowed with the following norm

$$\|f\|_{W^{1,1}(0,T;X)} = \|f\|_{L^1(0,T;X)} + \|f_t\|_{L^1(0,T;X)}.$$

2 Distributional Solutions for Small Initial Data

In order to apply the Kaniel and Shinbrot iteration technique we to introduce the (well known) trajectory operator #

$$f^\#(t, x, v) := f(t, x + tv, v).$$

This operator gives the evaluation along the trajectories of the transport operator $\partial_t + v \cdot \nabla$. Hence, (1.1) reduces to

$$\frac{df^\#}{dt}(t) = Q^\#(f, f)(t) \quad \text{with } f(0) = f_0. \tag{2.1}$$

Definition A *distributional solution* in $[0, T]$ of problem (1.1) is a function $f \in W^{1,1}(0, T; L^\infty(\mathbb{R}^{2n}))$ that solves (2.1) a.e. in $(0, T] \times \mathbb{R}^{2n}$.

Equation (2.1) is a good starting point to define the concept of solution because it does not requires differentiability in the x-variable for f, but equation (1.1) does. Moreover, if f is smooth in the x-variable equations (1.1) and (2.1) the notion of solutions are equivalent in the sense that f is a solution of the former if and only if is a solution of the later. In other words, (2.1) is a relaxed version of (1.1).

The concept of distributional (or *mild*) solution is suited to apply a technique introduced at the end of the 70's by Kaniel and Shinbrot [19]. This technique was first applied for these authors to find local in time mild solution for the Boltzmann equation. Later, it has been used with success to find global distributional solutions in the context of small initial data for the cases of elastic hard spheres in 3-dimension [18], relativistic Boltzmann [10], and recently the inelastic Boltzmann [2].

2.1 Kaniel and Shinbrot Iteration

In order to present Kaniel and Shinbrot technique we define the sequences $\{l_n(t)\}$ and $\{u_n(t)\}$ as the mild solutions of the linear problems

$$\frac{dl_n^\#}{dt}(t) + Q_-^\#(l_n, u_{n-1})(t) = Q_+^\#(l_{n-1}, l_{n-1})(t) \quad \text{and}$$

$$\frac{du_n^\#}{dt}(t) + Q_-^\#(u_n, l_{n-1})(t) = Q_+^\#(u_{n-1}, u_{n-1})(t), \tag{2.2}$$

with $0 \leq l_n(0) \leq f_0 \leq u_n(0)$. The construction begins by choosing a pair of functions (l_0, u_0) satisfying what Kaniel and Shinbrot called *the beginning condition* in $[0, T]$, i.e. $u_0^\# \in L^\infty(0, T; \mathcal{M}_{\alpha,\beta})$ and

$$0 \leq l_0^\#(t) \leq l_1^\#(t) \leq u_1^\#(t) \leq u_0^\#(t) \quad \text{a.e. in } 0 \leq t \leq T. \tag{2.3}$$

We summarize the results in [19] with the following theorem

Theorem 2.1 *Let the collision kernel B satisfy the assumptions* (A.1)–(A.2) *with condition* (A.1) *relaxed to* $-1 \leq \lambda < n - 1$, *and let* $\{l_n(t)\}$ *and* $\{u_n(t)\}$ *be the sequences defined by the mild solutions of the linear problems* (2.2). *In addition, assume that the beginning condition* (2.3) *is satisfied in* $[0, T]$, *then*

(i) *The sequences* $\{l_n(t)\}$ *and* $\{u_n(t)\}$ *are well defined for* $n \geq 1$. *In addition,* $\{l_n(t)\}$, $\{u_n(t)\}$ *are increasing and decreasing sequences respectively, and*

$$l_n^\#(t) \leq u_n^\#(t) \quad \text{a.e. in } 0 \leq t \leq T.$$

(ii) *If* $0 \leq l_n(0) = f_0 = u_n(0)$ *for* $n \geq 1$, *then*

$$\lim_{n \to \infty} l_n(t) = \lim_{n \to \infty} u_n(t) = f(t) \quad \text{a.e. in } [0, T].$$

The limit $f(t) \in C(0, T; \mathcal{M}_{\alpha,\beta})$ *is the unique distributional solution of the Boltzmann equation in* $[0, T]$ *and fulfills*

$$0 \leq l_0^\#(t) \leq f^\#(t) \leq u_0^\#(t) \quad \text{a.e. in } [0, T].$$

The following lemma, which holds for soft and hard potentials, provides the essential estimate used in the existence of distributional solutions in the near vacuum case.

Lemma 2.2 *Assume $-1 \leq \lambda < n - 1$. Then, for any $0 \leq s \leq t \leq T$ and functions $f^{\#}$, $g^{\#}$ that lie in $L^{\infty}(0, T; \mathcal{M}_{\alpha,\beta})$ the following inequality holds*

$$\int_s^t |Q_+^{\#}(f, g)(\tau)| \, d\tau \leq k_{\alpha,\beta} \exp(-\alpha|x|^2 - \beta|v|^2)$$

$$\times \|f^{\#}\|_{L^{\infty}(0,T;\mathcal{M}_{\alpha,\beta})} \|g^{\#}\|_{L^{\infty}(0,T;\mathcal{M}_{\alpha,\beta})}, \tag{2.4}$$

where

$$k_{\alpha,\beta} = \sqrt{\pi} \alpha^{-1/2} \|b\|_{L^1(S^{n-1})} \left(\frac{|S^{n-1}|}{n - \lambda - 1} + C_n \beta^{-n/2} \right),$$

with the constant C_n depending only on the dimension. In other words,

$$\int_0^T |Q_+^{\#}(f, g)(\tau)| \, d\tau \in L^{\infty}(0, T; \mathcal{M}_{\alpha,\beta}).$$

Proof An explicit calculation yields the inequality,

$$|Q_+^{\#}(f, g)(\tau, x, v)| \leq e^{-\beta|v|^2} \|f^{\#}\|_{L^{\infty}(0,T;\mathcal{M}_{\alpha,\beta})} \|g^{\#}\|_{L^{\infty}(0,T;\mathcal{M}_{\alpha,\beta})}$$

$$\times \int_{\mathbb{R}^n} e^{-\beta|v_*|^2} \int_{S^{n-1}} e^{-\alpha|x+\tau(v-'v)|^2 - \alpha|x+\tau(v-'v_*)|^2}$$

$$\times b(\hat{u} \cdot \sigma) \, d\sigma \, |u|^{-\lambda} \, dv_*. \tag{2.5}$$

Note that

$$|x + \tau(v - 'v)|^2 + |x + \tau(v - 'v_*)|^2 = |x|^2 + |x + \tau u|^2,$$

and,

$$\int_s^t e^{-\alpha|x+\tau u|^2} \, d\tau \leq \int_{-\infty}^{\infty} e^{-\alpha|\tau u|^2} \, d\tau \leq \frac{\sqrt{\pi}}{\alpha^{1/2}} |u|^{-1}.$$

Therefore, integrating (2.5) in $[s, t]$

$$\int_s^t |Q_+^{\#}(f, g)(\tau, x, v)| \, d\tau \leq \frac{\sqrt{\pi}}{\alpha^{1/2}} \|b\|_{L^1(S^{n-1})} \exp(-\alpha|x|^2 - \beta|v|^2)$$

$$\|f^{\#}\|_{L^{\infty}(0,T;\mathcal{M}^{\alpha,\beta})} \|g^{\#}\|_{L^{\infty}(0,T;\mathcal{M}^{\alpha,\beta})} \int_{\mathbb{R}^n} \exp(-\beta|v_*|^2)|u|^{-(\lambda+1)} \, dv_*.$$

Finally, the proof is completed by observing that,

$$\int_{\mathbb{R}^n} \exp(-\beta|v_*|^2)|u|^{-(\lambda+1)} \, dv_* \leq \int_{\{|v_*|<1\}} |u|^{-(\lambda+1)} \, dv_* + \int_{\{|v_*|\geq 1\}} \exp(-\beta|v_*|^2) \, dv_*$$

$$\leq \frac{|S^{n-1}|}{n - \lambda - 1} + C_n \beta^{-n/2}. \qquad \square$$

We are now in conditions to prove Theorem 2.3 for the global existence of distributional solutions for soft potentials. As we previously mentioned, this proof is valid for both soft and hard potentials as it relies solely on Lemma 2.2 and Theorem 2.1. The key step to apply Theorem 2.1 is to find suitable functions that satisfy the *beginning condition* globally. The most natural (and simplest) choice for the first members is

$$l_0^{\#} = 0 \quad \text{and} \quad u_0^{\#} = C \exp(-\alpha|x|^2 - \beta|v|^2).$$

Now compute the following two members

$$l_1^{\#}(t) = f_0 \exp\left(-\int_0^t R^{\#}(u_0)(\tau)\,d\tau\right) \quad \text{and} \quad u_1^{\#}(t) = f_0 + \int_0^t Q_+^{\#}(u_0, u_0)(\tau)\,d\tau.$$

Clearly, $0 \le l_0^{\#} \le l_1^{\#} \le u_1^{\#}$. In addition, using the previous expression and Lemma 2.2 we conclude that, for all $t \ge 0$,

$$u_1^{\#}(t) \le \left(\|f_0\|_{\alpha,\beta} + k_{\alpha,\beta}\|u_0^{\#}\|_{\alpha,\beta}^2\right) \exp(-\alpha|x|^2 - \beta|v|^2).$$

Noting that $\|u_0^{\#}\|_{\alpha,\beta} = C$, therefore it suffices to choose C as

$$\|f_0\|_{\alpha,\beta} + k_{\alpha,\beta}C^2 = C$$

in order to satisfy the beginning condition globally. This is actually possible for as long as

$$\|f_0\|_{\alpha,\beta} \le \frac{1}{4k_{\alpha,\beta}}.$$

In particular this previous calculation proves the following theorem.

Theorem 2.3 *Let $B(|u|, \hat{u} \cdot \sigma)$ satisfy the conditions (A.1)–(A.2) with condition (A.1) relaxed to $-1 \le \lambda < n - 1$. Then, the Cauchy-Boltzmann problem has a unique global distributional solution if*

$$\|f_0\|_{\alpha,\beta} \le \frac{1}{4k_{\alpha,\beta}},$$

where the constant $k_{\alpha,\beta}$ is given in Lemma 2.2. Moreover, such distributional solution satisfies

$$\|f^{\#}\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \le C := \frac{1 - \sqrt{1 - 4k_{\alpha,\beta}\|f_0\|_{\alpha,\beta}}}{2k_{\alpha,\beta}}, \tag{2.6}$$

for any $0 \le T \le \infty$.

As a consequence of Theorem 2.3, one concludes that the distributional solution f is controlled by a traveling Maxwellian, and that

$$\lim_{t \to \infty} f(t, x, \xi) \to 0 \quad \text{a.e. in } \mathbb{R}^{2n}.$$

3 Distributional Solutions Near Local Maxwellian

The aim of this section is to use Theorem 2.1 to construct solutions for the Cauchy Boltzmann problem in the soft potential case when the initial data is locally close to equilibrium; that is, the initial data is near to the local Maxwellian distribution given by (1.3). In contrast to the construction for small data made in the previous section, the negative part of the collision operator will be essential for this derivation. The main idea of this construction is taken from [24]. In addition, we refer to [21] that worked the Maxwellian case with infinite energy and to [11] for an elegant approach with an abstract fixed point argument.

Let us introduce some convenient notation and terminology that will help to maintain the proof short and clear. First, we define the *distance* between two Maxwellian distributions $M_i = C_i M_{\alpha_i, \beta_i}$ for $i = 1, 2$ as

$$d(M_1, M_2) := |C_2 - C_1| + |\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|.$$

Second, we say that f is ϵ-*close* to the Maxwellian distribution $M = C M_{\alpha, \beta}$ if there exist Maxwellian distributions M_i ($i = 1, 2$) such that $d(M_i, M) < \epsilon$ for some small $\epsilon > 0$, and

$$M_1 \leq f \leq M_2.$$

It is also convenient to define the function

$$\phi_{\alpha, \beta}(t, x, v) := \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp(-\alpha|x + u|^2 - \beta|v - u/t|^2)|u|^{-\lambda} \, du.$$

A simple analysis of this expression shows that the function $\phi_{\alpha, \beta}$ is bounded for $-n < \lambda \leq 0$ and that

$$\|\phi_{\alpha_2, \beta_2} - \phi_{\alpha_1, \beta_1}\|_{L^\infty} \leq C(\min \alpha_i, \min \beta_i) d(M_1, M_2), \tag{3.1}$$

for $0 < \alpha_i$ and $0 < \beta_i$. Similar control holds for $\beta_1 = \beta_2 = 0$ with constant depending only on $\min \alpha_i$.

Theorem 3.1 *Let $B(|u|, \hat{u} \cdot \sigma)$ satisfy the conditions (A.1) and (A.2). In addition, assume that $f_0(x, v)$ is ϵ-close to the local Maxwellian distribution $C M_{\alpha, \beta}(x - v, v)$ ($\alpha > 0$, $\beta > 0$, $C > 0$). Then, for sufficiently small ϵ depending on the initial parameters C, α, β, the model parameters $\lambda, \|b\|_{L^1(S^{n-1})}$ and dimension n, the Cauchy Boltzmann problem (1.1) has a unique solution satisfying*

$$C_1(t) M_{\alpha_1, \beta_1}(x - (t+1)v, v) \leq f(t, x, v) \leq C_2(t) M_{\alpha_2, \beta_2}(x - (t+1)v, v), \tag{3.2}$$

for all $t \geq 0$, for some continuous functions $0 < C_1(t) \leq C \leq C_2(t) < \infty$, and parameters $0 < \alpha_2 \leq \alpha \leq \alpha_1$ and $0 < \beta_2 \leq \beta \leq \beta_1$. Moreover, the case $\beta = 0$ (infinite mass) is permitted as long as $\beta_1 = \beta_2 = 0$.

Proof The idea of the proof is to solve the Cauchy Boltzmann problem for $t \geq 1$, with initial data f_0 given at $t = 1$. The translation (Galilean) invariance of the Boltzmann equation will imply the result for $t \geq 0$. To this end, we build a lower and upper barriers of the Kaniel and Shinbrot sequences and prove that they satisfy the *beginning condition*. Finally, we conclude the result using Theorem 2.1.

First note that since $f_0(x, v)$ is ϵ-*close* to the local Maxwellian $CM_{\alpha,\beta}(x - v, v)$, then, there exist two local Maxwellian distributions such that $d(M_i, M) < \epsilon$ $(i = 1, 2)$ and

$$C_1 M_{\alpha_1,\beta_1}(x, v) \leq f_0(x + v, v) \leq C_2 M_{\alpha_2,\beta_2}(x, v). \tag{3.3}$$

Since ϵ is expected to be small, we may assume that

$$\frac{1}{2}C \leq C_1 \leq C_2 \leq 2C, \qquad \frac{1}{2}\alpha \leq \alpha_2 \leq \alpha_1 \leq 2\alpha \quad \text{and} \quad \frac{1}{2}\beta \leq \beta_2 \leq \beta_1 \leq 2\beta. \tag{3.4}$$

Next, define the barriers $l_0^{\#}$ and $u_0^{\#}$ as follows:

$$0 \leq l_0^{\#}(t) = C_1(t)M_{\alpha_1,\beta_1} \quad \text{and} \quad u_0^{\#}(t) = C_2(t)M_{\alpha_2,\beta_2}. \tag{3.5}$$

The functions $C_1(t)$ and $C_2(t)$ will be chosen later satisfying

$$0 \leq C_1(t) \leq C_2(t) \quad \text{with } C_1(1) = C_1 \leq C_2 = C_2(1), \tag{3.6}$$

and that the following differential inequalities

$$\frac{dl_0^{\#}}{dt}(t) + Q_-^{\#}(l_0, u_0)(t) \leq Q_+^{\#}(l_0, l_0)(t) \quad \text{and} \tag{3.7}$$

$$\frac{du_0^{\#}}{dt}(t) + Q_-^{\#}(u_0, l_0)(t) \geq Q_+^{\#}(u_0, u_0)(t), \tag{3.8}$$

are satisfied for all $t \geq 1$. Assume for the moment that such functions $C_1(t)$ and $C_2(t)$ exist, then using the definition of $l_1^{\#}(t)$ and $u_1^{\#}(t)$ in the Kaniel–Shinbrot iteration (2.2)

$$\frac{dl_1^{\#}}{dt}(t) + Q_-^{\#}(l_1, u_0)(t) = Q_+^{\#}(l_0, l_0)(t), \tag{3.9}$$

$$\frac{du_1^{\#}}{dt}(t) + Q_-^{\#}(u_1, l_0)(t) = Q_+^{\#}(u_0, u_0)(t), \tag{3.10}$$

with initial condition

$$0 \leq C_1 M_{\alpha_1,\beta_1} = l_0^{\#}(1, x, v) \leq l_1^{\#}(1, x, v) = f_0(x + v, v)$$
$$= u_1^{\#}(1, x, v) \leq u_0^{\#}(1, x, v) = C_2 M_{\alpha_2,\beta_2}, \tag{3.11}$$

one obtains that $l_1^{\#}(t) \leq u_1^{\#}(t)$. Since $f_0(x + v, v) \geq C_1 M_{\alpha_1,\beta_1}$, we can use a classical comparison principle for ODE's between (3.7) and (3.9) to conclude that $l_0^{\#}(t) \leq l_1^{\#}(t)$. Similarly, $f_0(x + v, v) \leq C_2 M_{\alpha_1,\beta_1}$, thus, comparing (3.8) and (3.10) gives $u_1^{\#}(t) \leq u_0^{\#}(t)$. Therefore, the *beginning condition* holds for $t \geq 1$. An application of Theorem 2.1 in the interval $[1, \infty)$ and with initial condition $f_0(x + v, v)$ produces the desired solution $g(t, x, v)$ satisfying

$$C_1(t)M_{\alpha_1,\beta_1}(x, v) \leq g^{\#}(t, x, v) = g(t, x + tv, v) \leq C_2(t)M_{\alpha_1,\beta_1}(x, v) \quad \text{for } t \geq 1,$$

with $g^{\#}(1, x, v) = f_0(x + v, v) = g(1, x + v, v)$. This is precisely, up to a shift in time, the control (3.2). The proof concludes by just noticing that $f(t, x, v) := g(t + 1, x, v)$ solves the Boltzmann equation in $(0, \infty)$ with initial condition $f_0(x, v)$.

It remains to provide two functions $C_1(t)$ and $C_2(t)$ fulfilling conditions (3.6). First, note that a simple evaluation shows that right-hand side in (3.7) reduces to

$$Q_+^\#(l_0, l_0)(t) = C_1^2(t) M_{\alpha_1, \beta_1} \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp(-\alpha_1 |x + tu|^2 - \beta_1 |v - u|^2) |u|^{-\lambda} \, du$$

$$= \frac{C_1^2(t)}{t^{n-\lambda}} M_{\alpha_1, \beta_1} \phi_1,$$

where $\phi_1 := \phi_{\alpha_1, \beta_1}$. Similarly, the other term in (3.7) reads

$$Q_+^\#(l_0, u_0)(t) = C_1(t) C_2(t) M_{\alpha_1, \beta_1} \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp(-\alpha_2 |x + tu|^2 - \beta_2 |v - u|^2) |u|^{-\lambda} \, du$$

$$= \frac{C_1(t) C_2(t)}{t^{n-\lambda}} M_{\alpha_1, \beta_1} \phi_2,$$

where $\phi_2 := \phi_{\alpha_2, \beta_2}$. A similar calculation holds for the terms in (3.8).

Thus, the differential inequalities (3.7) and (3.8), after the cancellation of the Maxwellian distributions M_{α_i, β_i} $(i = 1, 2)$, are recast as

$$C_1'(t) \leq \frac{C_1^2(t)}{t^{n-\lambda}} \phi_1 - \frac{C_1(t) C_2(t)}{t^{n-\lambda}} \phi_2, \tag{3.12}$$

$$C_2'(t) \geq \frac{C_2^2(t)}{t^{n-\lambda}} \phi_2 - \frac{C_1(t) C_2(t)}{t^{n-\lambda}} \phi_1. \tag{3.13}$$

Next, in order to get uniform bounds in space, we want to get rid of the (x, v)-dependence in inequalities (3.12) and (3.13). Indeed, observe that the right hand side of the differential inequality (3.12) is uniformly controlled in (x, v) from below.

$$\frac{\phi_1 C_1^2(t) - \phi_2 C_1(t) C_2(t)}{t^{n-\lambda}} = \frac{C_1^2(t) - C_1(t) C_2(t)}{2 t^{n-\lambda}} (\phi_1 + \phi_2) + \frac{C_1^2(t) + C_1(t) C_2(t)}{2 t^{n-\lambda}} (\phi_1 - \phi_2)$$

$$\geq \frac{C_1^2(t) - C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 + \phi_2\|_{L^\infty}$$

$$- \frac{C_1^2(t) + C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 - \phi_2\|_{L^\infty}. \tag{3.14}$$

Similar calculations show that the right hand side of the differential inequality (3.13) is controlled by above

$$\frac{\phi_2 C_2^2(t) - \phi_1 C_1(t) C_2(t)}{t^{n-\lambda}} \leq \frac{C_2^2(t) - C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 + \phi_2\|_{L^\infty}$$

$$+ \frac{C_2^2(t) + C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 - \phi_2\|_{L^\infty}. \tag{3.15}$$

Therefore, we now are in conditions to find barrier functions $u_0^\#$ and $l_0^\#$ that satisfy (3.7) and (3.8) with the choice (3.5). In fact, a *sufficient condition* for functions $C_1(t)$ and $C_2(t)$ to satisfy inequalities (3.12) and (3.13), respectively, is obtain by choosing them as solutions of the following non-linear differential system

$$C_1'(t) = \frac{C_1^2(t) - C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 + \phi_2\|_{L^\infty} - \frac{C_1^2(t) + C_1(t) C_2(t)}{2 t^{n-\lambda}} \|\phi_1 - \phi_2\|_{L^\infty}, \tag{3.16}$$

$$C_2'(t) = \frac{C_2^2(t) - C_1(t)C_2(t)}{2t^{n-\lambda}}\|\phi_1 + \phi_2\|_{L^\infty} + \frac{C_2^2(t) + C_1(t)C_2(t)}{2t^{n-\lambda}}\|\phi_1 - \phi_2\|_{L^\infty}, \quad (3.17)$$

with initial condition $C_1(1) = C_1$ and $C_2(1) = C_2$.

Indeed, if $C_1(t)$ and $C_2(t)$ satisfy the differential system (3.16) and (3.17), the uniform control (3.14) and (3.15) readily gives that such functions satisfy the differential inequalities (3.12) and (3.13) as well.

In [21] the reader can find a proof, for $\lambda = 0$ and $\|\phi_1 - \phi_2\|_{L^\infty} = 0$ (i.e. $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$), that such $C_1(t)$ and $C_2(t)$ exist for $t \geq 1$ and are uniformly bounded on time provided that the parameter ϵ is sufficiently small. Here, we explicitly present the solution of such differential system by noticing that (3.16), (3.17) imply the simple differential relation

$$\frac{C_1'(t)}{C_1(t)} = -\frac{C_2'(t)}{C_2(t)},$$

which yields the algebraic relation for any $t \geq t_0 \geq 1$

$$\frac{C_1(t)}{C_1(t_0)} = \frac{C_2(t_0)}{C_2(t)}.$$

Hence, $C_2(t)$ obeys the equation

$$C_2'(t) + (\|\phi_1 + \phi_2\|_{L^\infty} - \|\phi_1 - \phi_2\|_{L^\infty}) \frac{C_1(1)C_2(1)}{2t^{n-\lambda}}$$

$$= (\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}) \frac{C_2^2(t)}{2t^{n-\lambda}},$$

which has explicit solutions.

Indeed, let

$$k^2 = \frac{\|\phi_1 + \phi_2\|_{L^\infty} - \|\phi_1 - \phi_2\|_{L^\infty}}{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}} C_1(1)C_2(1),$$

then

$$\frac{C_2(1) + k}{C_2(1) - k} \frac{C_2(t) - k}{C_2(t) + k} = \exp\left(k\frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\left(1 - \frac{1}{t^{n-\lambda-1}}\right)\right).$$

Therefore, $C_2(t)$ will remain uniformly bounded for $t \geq 1$ as long as

$$\exp\left(k\frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\right) < \frac{C_2(1) + k}{C_2(1) - k}. \quad (3.18)$$

Using (3.1) and (3.4), an elementary calculation shows that

$$|C_2(1) - k| \leq K_1(C, \alpha, \beta)d(M_1, M_2) \leq 2K_1(C, \alpha, \beta)\epsilon,$$

$$\exp\left(k\frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\right) \leq K_2(C, \alpha, \beta),$$

and

$$C_2(1) + k \geq K_3(C, \alpha, \beta).$$

Hence, inequality (3.18) is easily obtained for sufficiently small $\epsilon > 0$. Thus, the existence of continuous $C_1(t)$ and $C_2(t)$ are assured for $t \geq 1$. Moreover, as simple analysis of the previous expressions shows that $C_1(t)$ is non increasing and $C_2(t)$ is non decreasing, hence $C_1(t) \leq C_2(t)$, satisfying condition (3.6).

Finally, note that the norm $\|\phi_{\alpha,\beta}\|_{L^\infty}$ is, in general, controlled by a constant independent of β, hence by using (3.1) the case $\beta_1 = \beta_2 = 0$ of infinite mass and energy is included in the result. □

Remark In fact *item* (ii) of the original version of Theorem 2.1 was proved by Kaniel and Shinbrot assuming that $u_0^\#(t)$ was summable, however, this is not a major restriction for the result to be true as the authors in [21] proved.

4 Classical Solutions

We prove in this section the existence of classical solutions for the cases presented in the two previous ones. For this purpose we assume basic regularity in the initial data and then prove that such regularity is propagated through time. This approach is quite different to the one available by perturbation theory of a *global* Maxwellian (see [12, 13]), where high regularity in the initial data is crucial for their techniques based in Sobolev embedding theorems. Essentially, we follow the scheme presented in [6] adding some new features as needed.

Definition A *classical solution* in $[0, T]$ of problem (1.1) is a function such that

(i) $f(t) \in W^{1,1}(0, T; L^\infty(\mathbb{R}^{2n}))$
(ii) $\nabla_x f \in L^1(0, T; L^p(\mathbb{R}^{2n}))$ for some $1 \leq p$

which solves the Cauchy Boltzmann problem (1.1) a.e. in $[0, T) \times \mathbb{R}^{2n}$.

Before presenting the proof, we need the following estimate.

Theorem 4.1 *Let the collision kernel satisfy assumptions* (A.1) *and* (A.2) *in Sect.* 1.1. *Then,*

(i) *for* $1 < p, q, r < \infty$ *with* $1/p + 1/q + \lambda/n = 1 + 1/r$,

$$\|Q_\pm(f, g)\|_{L_v^r(\mathbb{R}^n)} \leq C_\pm \||u|^{-\lambda}\|_{L_w^{n/\lambda}(\mathbb{R}^n)} \|f\|_{L_v^p(\mathbb{R}^n)} \|g\|_{L_v^q(\mathbb{R}^n)}. \tag{4.1}$$

(ii) *Moreover, assume that*

$$B(|u|, \hat{u} \cdot \sigma) = \Phi(u) b(\hat{u} \cdot \sigma)$$

with $\Phi \in L^s(\mathbb{R}^n)$ *radially symmetric and non increasing. Then, for* $1 \leq p, q, r \leq \infty$ *with* $1/p + 1/q + 1/s = 1 + 1/r$ *one can estimate*

$$\|Q_\pm(f, g)\|_{L_v^r(\mathbb{R}^n)} \leq C_\pm \|\Phi\|_{L^s(\mathbb{R}^n)} \|f\|_{L_v^p(\mathbb{R}^n)} \|g\|_{L_v^q(\mathbb{R}^n)}, \tag{4.2}$$

where $C_\pm = C(n, p, q, r, \|b\|_{L^1(S^{n-1})})$ *for* $n \geq 3$.

The proof of these two estimates can be found in Sects. 4 and 5 of [4]. The constants are explicit

$$C_+ = C(n, p, q, r) \int_{-1}^{1} \left(\frac{2}{1-s} \right)^a b(s)(1-s^2)^{\frac{n-3}{2}} ds$$

and

$$C_- = C(n, p, q, r)\|b\|_{L^1(S^{n-1})},$$

for some positive constants $C(n, p, q, r)$ and a described in that reference. Note that there is a singularity at $s = 1$ in the integrand defining C_+. However, this singularity does not create any problem for the Boltzmann equation since the collision operator can be redefined, by symmetrization, to this new collision angular kernel

$$\bar{b}(s) := (b(s) + b(-s)) \chi\{s \leq 0\}.$$

This symmetrization yields the dependence of the constant C_+ on $\|b\|_{L^1(S^{n-1})}$ for $n \geq 3$.

Finally, observe that we have used the weak $L^{n/\lambda}$-norm of the soft potential $|u|^{-\lambda}$ in estimate (4.1). Also, for any $s > 1$ the *weak L^s-norm* of a function f is defined as

$$\|f\|_{L^s_w(\mathbb{R}^n)} = \sup_A |A|^{-1/s'} \int_A |f(x)| \, dx,$$

where $1/s + 1/s' = 1$ and A denotes an arbitrary measurable set of finite measure $|A| < \infty$.

Next, in order to study the propagation of regularity it is convenient to define for $h > 0$ and $\hat{x} \in S^{n-1}$ the following finite difference operator

$$(D_{h,\hat{x}} f)(x) := \frac{f(x + h\hat{x}) - f(x)}{h},$$

and also define the translation operator

$$(\tau_{h,\hat{x}} f)(x) := f(x + h\hat{x}).$$

For notation simplicity we write these operators as D and τ respectively.

Let f be the distributional solution of the Cauchy Boltzmann problem for either small data or near local Maxwellian case. Fix $h > 0$ and $\hat{x} \in S^{n-1}$ and apply the operator D to both sides of (2.1) to obtain

$$\frac{d(Df)^{\#}}{dt}(t) = (DQ(f, f))^{\#}(t) = Q^{\#}(Df, f)(t) + Q^{\#}(\tau f, Df)(t). \qquad (4.3)$$

Multiplying this equation by

$$p|(Df)^{\#}|^{p-1} \operatorname{sgn}((Df)^{\#}) \quad \text{for } p > 1,$$

and integrating in $\mathbb{R}^{2n}$ we are led to

$$\frac{d\|Df\|_{L^p}^p}{dt} = p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |Df|^{p-1} \operatorname{sgn}(Df) \left(Q(Df, f) + Q(\tau f, Df) \right) dv \, dx. \qquad (4.4)$$

Note first that $\operatorname{sgn}(Df) Q_-(Df, f) \leq 0$. Therefore, using (4.4) we obtain that for $1 < p < \infty$,

$$\frac{d\|Df\|_{L^p}^p}{dt} \leq p \int_{\mathbb{R}^n} \|Df\|_{L^p_v(\mathbb{R}^n)}^{p-1} (\|Q_+(Df, f)\|_{L^p_v(\mathbb{R}^n)} + \|Q_+(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)}$$

$$+ \|Q_-(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)}) \, dx$$

$$\leq pC \int_{\mathbb{R}^n} \|Df\|_{L_v^p(\mathbb{R}^n)}^p (\|f\|_{L_v^\gamma(\mathbb{R}^n)} + \|\tau f\|_{L_v^\gamma(\mathbb{R}^n)}) \, dx. \tag{4.5}$$

In obtaining these inequalities, we have used Hölder's inequality in the inner integral with exponent p and Theorem 4.1 with $(p, q, r) = (p, \gamma, p)$ and $(p, q, r) = (\gamma, p, p)$ where $\gamma := n/(n - \lambda)$. Moreover, the distributional solution $f(t, x, v)$ is controlled by a traveling Maxwellian, then

$$\|f\|_{L_v^\gamma(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n/\gamma}} = \frac{C}{(1+t)^{n-\lambda}},$$

and similarly

$$\|\tau f\|_{L_v^\gamma(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n-\lambda}}.$$

Notice that this estimate is valid for solutions with infinite mass and energy due to estimate (3.7). Thus, a Gronwall's argument on (4.5) shows that

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \leq \|Df_0\|_{L^p(\mathbb{R}^{2n})} \exp\left(\int_0^t \frac{C}{(1+s)^{n-\lambda}} \, ds\right), \tag{4.6}$$

where $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$.

Theorem 4.2 *Fix $0 \leq T \leq \infty$ and assume the conditions* (A.1)–(A.2) *in the collision kernel B. Also, assume that f_0 satisfies the smallness assumption of Theorem 2.3 or is near to a local Maxwellian as in Theorem 3.1. In addition, assume that $\nabla_x f_0 \in L^p(\mathbb{R}^{2n})$ for some $1 < p < \infty$. Then, there is a unique classical solution f to problem* (1.1) *in the interval $[0, T]$ satisfying the estimates of these theorems, and*

$$\|\nabla_x f\|_{L^p(\mathbb{R}^{2n})}(t) \leq C \|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})} \quad \text{for all } t \in [0, T], \tag{4.7}$$

with constant $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$.

Proof Thanks to Theorem 2.3 (or Theorem 3.1), we have existence of a unique distributional solution f to the Cauchy Boltzmann problem. Estimate (4.7) for $\nabla_x f$ follows after sending $h \to 0$ in inequality (4.6). After knowing that the weak gradient exist a.e. we can use the chain rule to obtain that for a.e. $(t, x, v) \in (0, T) \times \mathbb{R}^{2n}$

$$\frac{df^\#}{dt}(t, x, v) = \left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(t, x + tv, v) = Q(f, f)(t, x + tv, v).$$

Thus, f solves (1.1) a.e. in $(0, T) \times \mathbb{R}^{2n}$. $\square$

Remark We can also argue in the following way for the small initial data case: Impose Maxwellian decay on $\nabla_x f_0$ and observe that multiplying equation (4.3) by $\text{sgn}((Df)^\#(t))$, integrating the result in $(0, t)$ and using Lemma 2.2

$$\|(Df)^\#(t)\|_{\alpha,\beta} \leq \|Df_0\|_{\alpha,\beta} + k_{\alpha,\beta} \|(Df)^\#\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}$$
$$\times (\|f^\#\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} + \|(\tau f)^\#\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})})$$
$$\leq \|Df_0\|_{\alpha,\beta} + k_{\alpha,\beta} \|(Df)^\#\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \|f^\#\|_{L^\infty(0,T;\mathcal{M}_{2\alpha,\beta})} (1 + \exp(2\alpha h^2)).$$

Fix $\|f_0\|_{2\alpha,\beta} \le \frac{3}{16k_{2\alpha,\beta}}$ and use Theorem 2.3 to obtain that the distributional solution fulfills the estimate

$$\|f^{\#}\|_{L^\infty(0,T;\mathcal{M}_{2\alpha,\beta})} \le \frac{1}{4k_{2\alpha,\beta}}.$$

But $\frac{k_{\alpha,\beta}}{k_{2\alpha,\beta}} = \sqrt{2}$, therefore,

$$\|(Df)^{\#}\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \le \frac{4\|Df_0\|_{\alpha,\beta}}{4 - \sqrt{2}(1 + \exp(2\alpha h^2))}.$$

Send $h \to 0$ to conclude that the distributional solution satisfies

$$\|(\nabla_x f)^{\#}\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \le \frac{2\|\nabla_x f_0\|_{\alpha,\beta}}{2 - \sqrt{2}}.$$

This procedure yields a classical solution to problem (1.1) with Maxwellian decay in its gradient provided small initial data. Thus, an iterative use of the argument above can produce classical solutions as smooth as desired on the condition that the initial datum is smooth with its derivatives having Maxwellian decay. Observe that the process is valid for both soft and hard potentials and only uses the integrability of b.

4.1 Velocity Regularity

In this subsection we investigate briefly the propagation of velocity smoothness for the classical solutions obtained above. To this end we use the finite difference operator for the v-variable

$$(D_{h,\hat{v}} f)(v) := \frac{f(v + h\hat{v}) - f(v)}{h},$$

for a fix $h > 0$ and $\hat{v} \in S^{n-1}$. Similarly for the translation operator $\tau_{h,\hat{v}}$. As above, we write such operators as D and τ for notation simplicity. Take a classical solution o f the Cauchy Boltzmann problem f and apply the finite difference operator in (1.1) to obtain

$$\frac{d(Df)}{dt}(t) + v \cdot \nabla_x(Df)(t) + \hat{v} \cdot \nabla_x(\tau f)(t)$$
$$= DQ(f, f)(t) = Q(Df, f)(t) + Q(\tau f, Df)(t).$$

This equality follows after using the change of variables $v_* \to v_* + h\hat{v}$ in the collision operator. Moreover, multiplying this equation by

$$p|(Df)|^{p-1}\,\mathrm{sgn}((Df)) \quad \text{for } p > 1,$$

and integrating in $\mathbb{R}^{2n}$, we obtain

$$\frac{d\|Df\|_{L^p}^p}{dt}(t) \le p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |Df|^{p-1}\mathrm{sgn}(Df)(Q(Df, f) + Q(\tau f, Df))\,dv\,dx$$
$$+ p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla_x(\tau f)||Df|^{p-1}\,dv\,dx$$
$$\le pC \int_{\mathbb{R}^n} \|Df\|_{L_v^p(\mathbb{R}^n)}^p (\|f\|_{L_v^\gamma(\mathbb{R}^n)} + \|\tau f\|_{L_v^\gamma(\mathbb{R}^n)})\,dx$$

$$+ p\|Df\|_{L^p(\mathbb{R}^{2n})}^{p-1}\|\nabla_x(\tau f)\|_{L^p(\mathbb{R}^{2n})}$$

$$\leq \frac{pC}{(1+t)^{n-\lambda}}\|Df\|_{L^p(\mathbb{R}^{2n})}^p + p\|Df\|_{L^p(\mathbb{R}^{2n})}^{p-1}\|\nabla_x f\|_{L^p(\mathbb{R}^{2n})}. \tag{4.8}$$

In order to quantify the size of the L^p-norm of Df let $X(t) := \|Df\|_{L^p(\mathbb{R}^{2n})}^p(t)$, then inequality (4.8) takes the classical non-linear Ordinary Differential inequality form (Bernoulli type)

$$\frac{dX(t)}{dt} \leq \mathsf{a}(t)X(t) + \mathsf{b}(t)X^{\frac{p-1}{p}}(t),$$

with

$$\mathsf{a}(t) = \frac{pC}{(1+t)^{n-\lambda}} \quad \text{and} \quad \mathsf{b}(t) = p\|(\nabla_x f)(t)\|_{L^p(\mathbb{R}^{2n})}^{p-1}.$$

Therefore,

$$X^{\frac{1}{p}}(t) \leq X_0^{\frac{1}{p}} \exp\left(\frac{1}{p}\int_0^t \mathsf{a}(s)\,ds\right) + \frac{1}{p}\int_0^t \exp\left(\frac{1}{p}\int_r^t \mathsf{a}(s)\,ds\right)\mathsf{b}(r)\,dr,$$

hence, using estimate (4.7)

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \leq (\|Df_0\|_{L^p(\mathbb{R}^{2n})} + t\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})})\exp\left(\int_0^t \frac{C}{1+s^{n-\lambda}}\,ds\right).$$

Thus, letting $h \to 0$ we conclude the following theorem.

Theorem 4.3 *Let f be a classical solution in $[0, T]$ with f_0 satisfying the condition of Theorem 2.3 or Theorem 3.1 and $\nabla_x f_0 \in L^p(\mathbb{R}^{2n})$ for some $1 < p < \infty$. In addition assume that $\nabla_v f_0 \in L^p(\mathbb{R}^{2n})$. Then, f satisfies the estimate*

$$\|(\nabla_v f)(t)\|_{L^p(\mathbb{R}^{2n})} \leq C(\|\nabla_v f_0\|_{L^p(\mathbb{R}^{2n})} + t\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}), \tag{4.9}$$

with $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$ independent of time.

4.2 L^p and $\mathcal{M}_{\alpha,\beta}$ stability

We finish this last section by presenting a short discussion on the stability of solutions in the L^p ($1 \leq p \leq \infty$) and $\mathcal{M}_{\alpha,\beta}$ spaces also for S^{n-1} integrability in the angular part $b(\hat{u}\cdot\sigma)$ of the collision kernel (Grad cut-off assumption). We refer to [14] and [15] for a discussion on the L^1 stability for solutions near vacuum and near Maxwellian for soft and hard potentials under the pointwise condition on the angular part of the collision kernel given by $b(\hat{u}\cdot\sigma) \leq K\cos(\hat{u}\cdot\sigma)$.

First, take f and g solutions of the Boltzmann Cauchy problem associated to the initial data f_0 and g_0 respectively. These data fulfill the condition of Theorem 2.3 (or Theorem 3.1) so that f and g are controlled by Maxwellian distributions as described in these theorems. Thus,

$$\frac{d(f-g)^{\#}}{dt}(t) = Q^{\#}(f, f)(t) - Q^{\#}(g, g)(t) = \frac{1}{2}[Q^{\#}(f-g, f+g) - Q^{\#}(f+g, f-g)].$$

After multiplying by $|(f - g)^{\#}|^{p-1}\mathrm{sgn}((f - g)^{\#})$ with $1 < p < \infty$ and following the usual steps we arrive to

$$\frac{d\|f - g\|_{L^p}^p}{dt}(t) \leq C \int_{\mathbb{R}^n} \|f - g\|_{L_v^p(\mathbb{R}^n)}^p \|f + g\|_{L_v^\gamma(\mathbb{R}^n)} \, dx.$$

Since f and g are controlled by traveling Maxwellian distributions

$$\|f + g\|_{L_v^\gamma(\mathbb{R}^n)} \leq \frac{C}{(1 + t)^{n-\lambda}}. \tag{4.10}$$

Therefore, an application of Gronwall's lemma leads to the following theorem.

Theorem 4.4 *Let f and g distributional solutions of problem* (1.1) *associated to the initial data f_0 and g_0 respectively. Assume that these data satisfy the condition of Theorem* 2.3 (*or Theorem* 3.1). *Then, there exist a constant $C > 0$ independent of time such that*

$$\|f - g\|_{L^p(\mathbb{R}^{2n})} \leq C \|f_0 - g_0\|_{L^p(\mathbb{R}^{2n})} \quad \text{with } 1 \leq p \leq \infty. \tag{4.11}$$

Moreover, for f_0 and g_0 sufficiently small in $\mathcal{M}_{\alpha,\beta}$ it holds

$$\|(f - g)^{\#}\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq C \|f_0 - g_0\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}. \tag{4.12}$$

Proof The argument preceding (4.10) shows the theorem holds for $p \in (1, \infty)$. Thus it remains to prove estimate (4.11) just for the cases $p = 1$ or $p = \infty$.

To this end we split the soft potential in two radially symmetric and decreasing potentials

$$|u|^{-\lambda} = \Phi_1(u) + \Phi_2(u),$$

where $\Phi_1(u) \in L^s(\mathbb{R}^n)$ for any $1 \leq s < n/\lambda$ and $\Phi_2(u) \in L^\infty$. For instance,

$$\Phi_1(u) = (|u|^{-\lambda} - 1)\chi_{\{|u| \leq 1\}} \quad \text{and} \quad \Phi_2(u) = |u|^{-\lambda} - \Phi_1.$$

Then, using (4.2) in Theorem 4.1 with $(p, q, s, r) = (1, s', s, 1)$ and $(p, q, s, r) = (\infty, s', s, \infty)$ we have for any $1 \leq s < n/\lambda$

$$\|Q(f - g, f + g)\|_{L_v^p(\mathbb{R}^n)} \leq \|Q_{\Phi_1}(f - g, f + g)\|_{L_v^1(\mathbb{R}^n)} + \|Q_{\Phi_2}(f - g, f + g)\|_{L_v^p(\mathbb{R}^n)}$$

$$\leq C_s \|\Phi_1\|_{L_v^s(\mathbb{R}^n)} \|f + g\|_{L_v^{s'}(\mathbb{R}^n)} \|f - g\|_{L_v^p(\mathbb{R}^n)}$$

$$+ C \|\Phi_2\|_{L_v^\infty(\mathbb{R}^n)} \|f + g\|_{L_v^1(\mathbb{R}^n)} \|f - g\|_{L_v^p(\mathbb{R}^n)},$$

where $1/s + 1/s' = 1$. Using the control by traveling Maxwellians we have

$$\|f + g\|_{L_v^{s'}(\mathbb{R}^n)} \leq \frac{C}{(1 + t)^{n/s'}} \quad \text{and} \quad \|f + g\|_{L_v^1(\mathbb{R}^n)} \leq \frac{C}{(1 + t)^n}.$$

Hence, after previously described computations, we obtain the estimate

$$\|f - g\|_{L^p}(t) \leq \|f_0 - g_0\|_{L^p} + \int_0^t \frac{C_s}{(1 + \tau)^{n/s'}} \|f - g\|_{L^p(\mathbb{R}^{2n})}(\tau) \, d\tau.$$

Choosing any $s \in (\frac{n}{n-1}, \frac{n}{\lambda})$ one has that $n/s' > 1$. Therefore Gronwall's lemma implies that

$$\|f - g\|_{L^p(\mathbb{R}^{2n})} \leq C \|f_0 - g_0\|_{L^p(\mathbb{R}^{2n})}$$

with C independent of time.

The proof of estimate (4.12) is a direct consequence of Lemma 2.2 and estimate (2.6). □

Remark Clearly, Theorem 4.4 gives uniqueness of solutions $f^\# \in L^\infty(0, T; \mathcal{M}_{\alpha,\beta})$ (for $\alpha > 0, \beta > 0$). So, in particular, solutions constructed in [11] are unique.

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